Finding Optimal Formulae for Bilinear Maps

Razvan Barbulescu, Jérémie Detrey, Nicolas Estibals, Paul Zimmermann

Arén^H^H^HRIC seminar, LIP — March 22, 2012

Finding Optimal Formulae for Bilinear Maps

(slides courtesy of Nicolas Estibals)

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A bit of history

- Multiplication is an expensive arithmetic operation
- Well-studied problem
  - Karatsuba (1962)
  - Toom–Cook (1963), evaluation-interpolation schemes
  - CRT-based algorithms
  - Schönhage–Strassen algorithm (1971)
  - ...
- *Five-, six-, and seven-term Karatsuba-like formulae*, P. Montgomery (2005)
  - ad-hoc formulae
  - exhaustive search for five-term multiplication
  - non-exhaustive search for six- and seven-term multiplications
  - (January 2011) start a task group to reproduce this work

Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

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Example: a 2-term polynomial times a 3-term one

- We want to compute

$$\begin{aligned} C(X) &= (a_1 \cdot X + a_0) \times (b_2 \cdot X^2 + b_1 \cdot X + b_0) \\ &= a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + (a_0 b_1 + a_1 b_0) \cdot X + a_0 b_0 \end{aligned}$$

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- Only 5 products required instead of 6
  - use Karatsuba's trick

$$C(X) = a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + ((a_0 + a_1)(b_0 + b_1) - a_1 b_1 - a_0 b_0) \cdot X + a_0 b_0$$

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  - compute the products

$$\begin{aligned} g_0 &= a_0 \cdot b_0, \\ g_1 &= a_0 \cdot b_2, \\ g_2 &= a_1 \cdot b_1, \\ g_3 &= a_1 \cdot b_2, \text{ and} \\ g_4 &= (a_0 + a_1) \cdot (b_0 + b_1). \end{aligned}$$

  - reconstruct the result

$$C(X) = g_3 \cdot X^3 + (g_1 + g_2) \cdot X^2 + (g_4 - g_2 - g_0) \cdot X + g_0$$

General form of a multiplication formula

- We want to compute, over a given field K (or any K-algebra $\mathbf{K}$),
  $$(a_{n-1} \cdot X^{n-1} + \cdots + a_0) \times (b_{m-1} \cdot X^{m-1} + \cdots + b_0) = c_{n+m-2} \cdot X^{n+m-2} + \cdots + c_0$$

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  - reconstruct the result by linear combinations of the products
    $$c_k = \sum \gamma_{j,k} \cdot g_j, \quad \text{with } \gamma_{j,k} \in K$$
- This is also valid for any bilinear map
  $$\begin{array}{ccccccc} F: & K^n & \times & K^m & \longrightarrow & K^\ell \\ & ((a_0, \ldots, a_{n-1}) & , & (b_0, \ldots, b_{m-1})) & \longmapsto & (c_0, \ldots, c_{\ell-1}) \end{array}$$

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$$F : K^n \times K^m \to K^\ell$$

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- Generators: the set $\mathcal{G} \subset V$ of the potential products to use in a formula

$$\mathcal{G} = \{(\textstyle\sum \alpha_i a_i) \cdot (\sum \beta_j b_j) \mid \forall i,\ \alpha_i \in K \text{ and } \forall j,\ \beta_j \in K\} \setminus \{0\}$$

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We only consider products modulo a nonzero scalar factor

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where $g \sim g'$ when $\exists \lambda \in K$, $\lambda \neq 0$ such that $g = \lambda g'$

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- The set of generators contains 21 products:

$$\begin{aligned}
\mathcal{G} = \{\ & a_0 \cdot b_0, & & a_1 \cdot b_0, & & (a_0 + a_1) \cdot b_0, \\
& a_0 \cdot b_1, & & a_1 \cdot b_1, & & (a_0 + a_1) \cdot b_1, \\
& a_0 \cdot (b_0 + b_1), & & a_1 \cdot (b_0 + b_1), & & (a_0 + a_1) \cdot (b_0 + b_1), \\
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& a_0 \cdot (b_0 + b_1 + b_2), & & a_1 \cdot (b_0 + b_1 + b_2), & & (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}
\end{aligned}$$

Naive algorithm

- Goal: find the optimal formulae (*i.e.*, with a minimum number of products)
  - enumerate the subsets $\mathcal{W} \subset \mathcal{G}$ of exactly k products which yield a valid formula
  - starting with $k = \mathrm{rk}(\mathcal{T})$, increase k until a solution is found

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- Drawback: Distinct subsets may span the same subspace

Improved algorithm: subspaces instead of subsets

- Look instead for subspaces W of V such that
  - W can be generated by products only: $\mathrm{Span}\,(W \cap \mathcal{G}) = W$
  - only k products are required: $\dim W = k$
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- Algorithm:

procedure `expand_subspace`(W)
    if $\dim W = k$ **and** $T \subset W$ **then**
        W is a solution
    else if $\dim W < k$ **then**
        for each $g \in \mathcal{G} \setminus W$ **do**
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- Several formulae can correspond to a single solution subspace W
  - each basis of W comprising only elements of $\mathcal{G}$ gives a k-product formula

Improved algorithm: incomplete basis

- We already know part of W!
  - target space T is a subspace of every solution space W
  - find each W by constructing $\mathcal{W}' \subset \mathcal{G}$ such that $W = T \oplus \operatorname{Span} \mathcal{W}'$

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procedure expand_subspace(W)
    if dim W = k and T ⊂ W then
        W is a solution
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        for each g ∈ 𝒢 \ W do
            expand_subspace(W ⊕ Span(g))
end procedure
expand_subspace({0})
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- Complexity now depends on

$$\binom{\#\mathcal{G}}{k - \operatorname{rk} \mathcal{T}}$$

Example (cont'd)

2×3-term polynomial product in $\mathbb{F}_2[X]$

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- Attempt with $k = 4$:
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  - $\mathrm{rk}(T \cap \mathcal{G}) = 3 < k$
  - no solutions with $k = 4$ products only

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  - at least $\mathrm{rk}(\mathcal{T}) = 4$ products required
- Attempt with $k = 4$:
  - $W = T$
  - $T \cap \mathcal{G} = \{\ a_0 \cdot b_0,\ \ a_1 \cdot b_2,\ \ (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}$
  - $\mathrm{rk}(T \cap \mathcal{G}) = 3 < k$
  - no solutions with $k = 4$ products only
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  - there are 3 such solution spaces which yield a total of 162 formulae

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  - example: squaring of a 2-term polynomial in $\mathbb{F}_3[X]$

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Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

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- Going further: instead of a single $\sigma \in \mathrm{Aut}(V)$, take a whole subgroup $S \subset \mathrm{Aut}(V)$

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  - $\sigma_0 : a_i b_j \mapsto a_{1-i} b_{2-j}$ (*i.e.*, replace X by $1/X$)
  - $\sigma_1 : a_i b_j \mapsto \left(\sum_{k=i}^{1} \binom{k}{i} a_k \right) \cdot \left(\sum_{k=j}^{2} \binom{k}{j} b_k \right)$ (*i.e.*, replace X by $1 - X$)
- Remark that $\#S = 6$ (hurray for non-commutativity!)
- $\mathcal{G}$ can be partitioned into
  - 3 orbits of size 3:

$$\begin{aligned}
\overline{g_0} &= \{\ a_0 \cdot b_0, & a_1 \cdot b_2, & \quad (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}, \\
\overline{g_1} &= \{\ a_0 \cdot b_1, & a_1 \cdot b_1, & \quad (a_0 + a_1) \cdot b_1\ \}, \\
\overline{g_2} &= \{\ a_0 \cdot (b_0 + b_1), & a_1 \cdot (b_1 + b_2), & \quad (a_0 + a_1) \cdot (b_0 + b_2)\ \}, \text{ and}
\end{aligned}$$

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

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\end{aligned}$$

  - 2 orbits of size 6:

$$\begin{aligned}
\overline{g_3} = \{\ & a_0 \cdot b_2, && a_0 \cdot (b_0 + b_1 + b_2), && (a_0 + a_1) \cdot b_0,\\
& a_1 \cdot b_0, && a_1 \cdot (b_0 + b_1 + b_2), && (a_0 + a_1) \cdot b_2\ \},\\
\overline{g_4} = \{\ & a_0 \cdot (b_0 + b_2), && a_0 \cdot (b_1 + b_2), && (a_0 + a_1) \cdot (b_0 + b_1),\\
& a_1 \cdot (b_0 + b_2), && a_1 \cdot (b_0 + b_1), && (a_0 + a_1) \cdot (b_1 + b_2)\ \}
\end{aligned}$$

Orbits of a group action: Example

2 × 3-term polynomial product in $\mathbb{F}_2[X]$

- First, consider adding to T a generator from orbit $\overline{g_1}$:

$$\sigma_0 \circlearrowright (a_0 + a_1) \cdot b_1 \overset{\sigma_1}{\leftrightarrows} a_0 \cdot b_1 \overset{\sigma_0}{\leftrightarrows} a_1 \cdot b_1 \circlearrowleft \sigma_1$$

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T

Orbits of a group action: Example

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T

$\cdots$

$(a_0 + a_1) \cdot b_1$

$a_0 \cdot b_1$

$a_1 \cdot b_1$

$\cdots$

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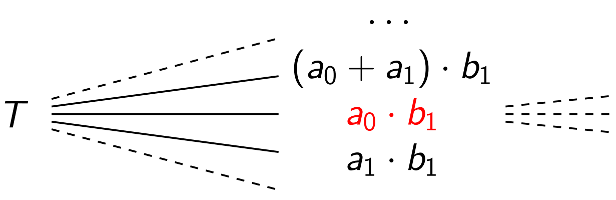

Orbits of a group action: Example

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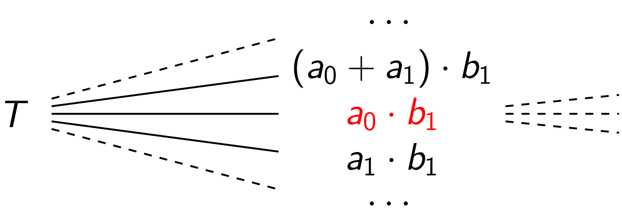
- For any solution of the form $W = T \oplus \mathrm{Span}(a_0 \cdot b_1) \oplus \mathrm{Span}\,\mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:



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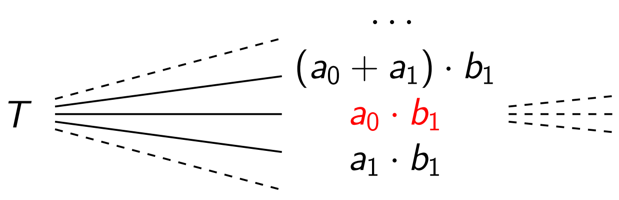

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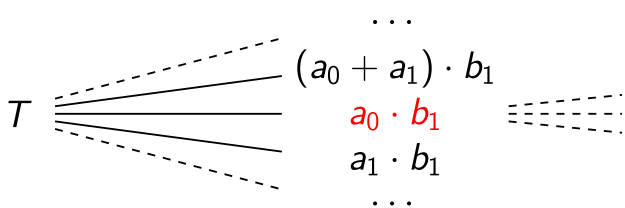
  - thus, no need to enumerate both $(a_0 + a_1) \cdot b_1$ and $a_0 \cdot b_1$



Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- First, consider adding to T a generator from orbit $\overline{g_1}$:

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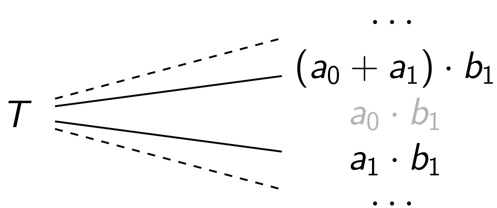

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- First, consider adding to T a generator from orbit $\overline{g_1}$:

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- For any solution of the form $W = T \oplus \operatorname{Span}(a_0 \cdot b_1) \oplus \operatorname{Span} \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:
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- Similarly, for any solution $W = T \oplus \operatorname{Span}(a_1 \cdot b_1) \oplus \operatorname{Span} \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:
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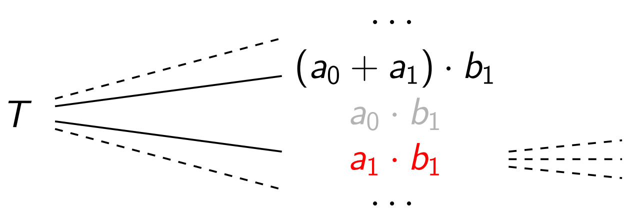

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- So we add generator $(a_0 + a_1) \cdot b_1$, fixed by σ_0

$$T \quad \begin{array}{c} \cdots \\ (a_0 + a_1) \cdot b_1 \\ a_0 \cdot b_1 \\ a_1 \cdot b_1 \\ \cdots \end{array}$$

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- So we add generator $(a_0 + a_1) \cdot b_1$, fixed by σ_0
- Now, consider adding to $T \oplus \mathrm{Span}((a_0 + a_1) \cdot b_1)$ a generator from orbit $\overline{g_3}$:

$$\begin{tikzcd}
a_0 \cdot b_2 \arrow[r, leftrightarrows, "\sigma_0"] \arrow[d, shift left, "\sigma_1"'] & a_1 \cdot b_0 \arrow[r, leftrightarrows, "\sigma_1"] & a_1 \cdot (b_0 + b_1 + b_2) \arrow[d, shift left, "\sigma_0"] \\
(a_0 + a_1) \cdot b_2 \arrow[u, shift left] \arrow[r, leftrightarrows, "\sigma_0"] & (a_0 + a_1) \cdot b_0 \arrow[r, leftrightarrows, "\sigma_1"] & a_0 \cdot (b_0 + b_1 + b_2) \arrow[u, shift left]
\end{tikzcd}$$

T

$\dots$

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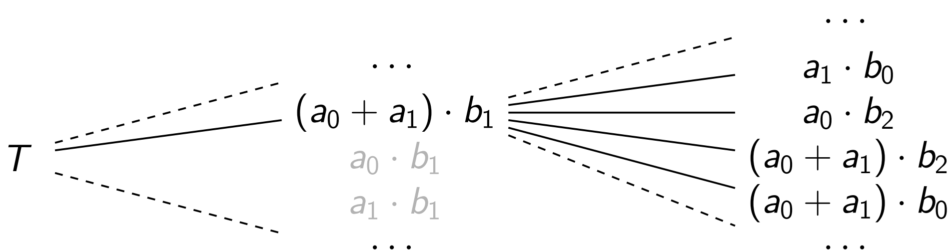

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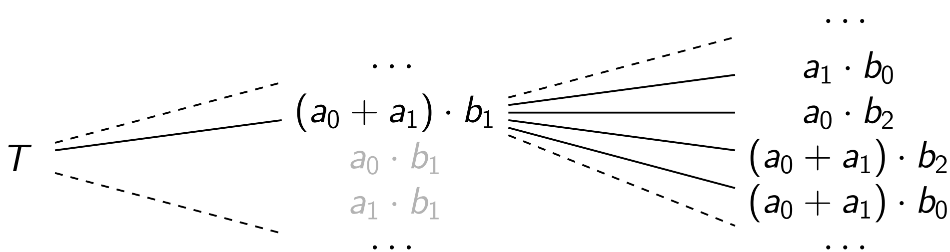
- Since $T \oplus \mathrm{Span}((a_0 + a_1) \cdot b_1)$ is fixed by σ_0, we can still partition $\overline{g_3}$ according to its σ_0-orbits



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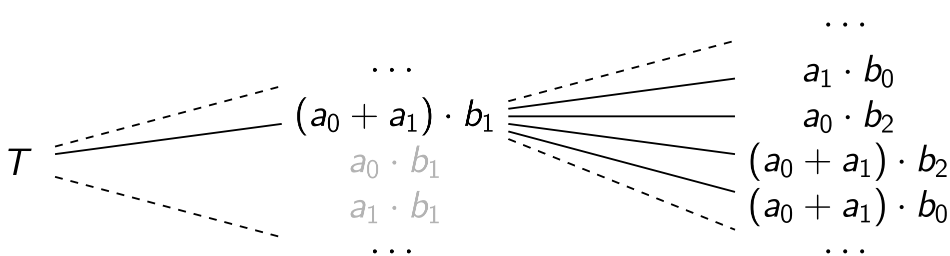

Orbits of a group action: Example

2 × 3-term polynomial product in $\mathbb{F}_2[X]$

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$$
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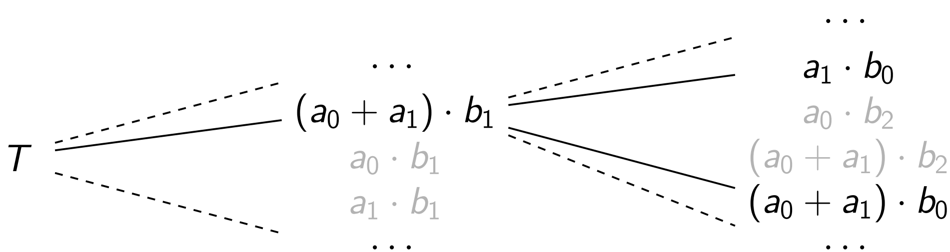

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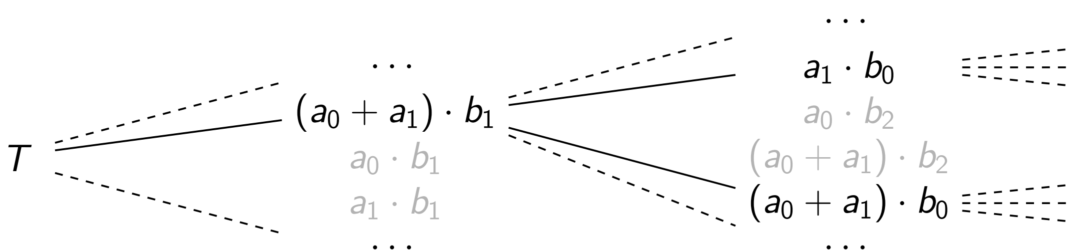

Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

n × m-term polynomial multiplication

Ring	n × m		#$\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$F_2[X]$	2 × 2		9	3	1	1	1	0.00
	3 × 3		49	6	9	3	9	0.00
	4 × 4		225	9	$6.60 \cdot 10^3$	4	4	0.03
	5 × 5		961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
	6 × 6		3 969	14	$4.37 \cdot 10^9$	—	—	$6.03 \cdot 10^5$
		(Sym.)	63	17	$8.08 \cdot 10^6$	6	54	17.7
	7 × 7	(Sym.)	127	22	$3.38 \cdot 10^{12}$	2 618	19 550	$1.59 \cdot 10^7$
$F_3[X]$	2 × 2		16	3	1	1	4	0.00
	3 × 3		169	6	24	22	1 493	0.00
	4 × 4		1 600	9	$4.11 \cdot 10^5$	726	50 640	14.9
	5 × 5		14 641	11	$4.89 \cdot 10^7$	—	—	$4.02 \cdot 10^4$
		(Sym.)	121	12	$3.93 \cdot 10^4$	31	6 460	0.14
	6 × 6	(Sym.)	364	15	$2.37 \cdot 10^8$	4	1 024	$3.79 \cdot 10^3$
	7 × 7	(Sym.)	1 093	17	$2.69 \cdot 10^{10}$	—	—	$1.50 \cdot 10^6$

$n \times m$-term polynomial multiplication

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	4 × 4	225	9	$6.60 \cdot 10^3$	4	4	0.03
	5 × 5	961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
	6 × 6	3 969	14	$4.37 \cdot 10^9$	—	—	$6.03 \cdot 10^5$
		(Sym.) 63	17	$8.08 \cdot 10^6$	6	54	17.7
	7 × 7	(Sym.) 127	22	$3.38 \cdot 10^{12}$	2 618	19 550	$1.59 \cdot 10^7$
$F_3[X]$	2 × 2	16	3	1	1	4	0.00
	3 × 3	169	6	24	22	1 493	0.00
	4 × 4	1 600	9	$4.11 \cdot 10^5$	726	50 640	14.9
	5 × 5	14 641	11	$4.89 \cdot 10^7$	—	—	$4.02 \cdot 10^4$
		(Sym.) 121	12	$3.93 \cdot 10^4$	31	6 460	0.14
	6 × 6	(Sym.) 364	15	$2.37 \cdot 10^8$	4	1 024	$3.79 \cdot 10^3$
	7 × 7	(Sym.) 1 093	17	$2.69 \cdot 10^{10}$	—	—	$1.50 \cdot 10^6$

$n \times m$-term polynomial multiplication

Ring	$n \times m$	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$F_2[X]$	2×2	9	3	1	1	1	0.00
	3×3	49	6	9	3	9	0.00
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Multiplication in small finite-field extensions

Finite field	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$\mathbf{F}_{2^2}$	9	3	3	3	3	0.00
$\mathbf{F}_{2^3}$	49	6	$7.03 \cdot 10^3$	105	147	0.01
$\mathbf{F}_{2^4}$	225	9	$2.57 \cdot 10^9$	2 025	2 025	$1.13 \cdot 10^4$
$\mathbf{F}_{2^5}$	961	9	$3.10 \cdot 10^{10}$	—	—	$8.11 \cdot 10^5$
	(Sym.) 31	13	$3.49 \cdot 10^6$	2 015	2 015	6.24
$\mathbf{F}_{2^6}$	(Sym.) 63	15	$2.21 \cdot 10^{10}$	21	21	$6.63 \cdot 10^4$
$\mathbf{F}_{2^7}$	(Sym.) 127	15	$1.34 \cdot 10^{12}$	—	—	$6.17 \cdot 10^6$
$\mathbf{F}_{3^2}$	16	3	3	4	16	0.00
$\mathbf{F}_{3^3}$	169	6	$2.42 \cdot 10^5$	11 843	105 963	1.08
$\mathbf{F}_{3^4}$	1 600	8	$2.27 \cdot 10^{11}$	—	—	$1.08 \cdot 10^7$
	(Sym.) 40	9	$1.10 \cdot 10^5$	234	615 240	0.45
$\mathbf{F}_{3^5}$	(Sym.) 121	11	$2.66 \cdot 10^9$	121	121	$1.45 \cdot 10^4$
$\mathbf{F}_{3^6}$	(Sym.) 364	12	$3.01 \cdot 10^{12}$	—	—	$4.50 \cdot 10^7$

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  - We can cherry-pick the one with minimum number of additions and scalar multiplications
- Work in progress and perspectives
  - Lifting formulae for higher-characteristic or characteristic-0 fields
  - Find formulae for your bilinear application!

Thank you for your attention

Questions?

Representing elements

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&= \begin{array}{c} \begin{array}{cccccc} a_0 b_0 & a_0 b_1 & a_0 b_2 & a_1 b_0 & a_1 b_1 & a_1 b_2 \end{array} \\ (\; 0, \;\; 1, \;\; 0, \; -1, \;\; 0, \;\; 1 \;) \end{array}
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uintN_t v[2] = { 0x22u, 0x08u };
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&= \quad (\; 0, \; 1, \; 0, \; 2, \; 0, \; 1 \;) \\
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    e.g., addition $w \leftarrow u + v$ with $K = \mathbb{F}_3$:

```
w[0] = (u[0]|v[0]) ^ ((u[0]|u[1]) & (v[0]|v[1]));
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```

  - machine-specific bound on N: ≤ 64, ≤ 128 (SSE2), or ≤ 256 (AVX)

Representing subspaces

- Subspaces $W \subset V$ represented using matrices in row echelon form

$$W = \mathrm{RowSpace} \left. \overbrace{\begin{pmatrix} 0 & w_{1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & w_{2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & 0 & 0 & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{r,j_r} & \cdots \end{pmatrix}}^{\dim V = N \text{ columns}} \right\} \dim W = r \text{ rows}$$

where each pivot $w_{i,j_i} \neq 0$, and with $j_1 < j_2 < \cdots < j_r$

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0 & 0 & 0 & w_{2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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  - ☹ waste of memory
  - ☺ easier to insert extra rows when expanding W

Reduction by a subspace

- Given a vector $v \in V$ and a subspace W:
  - sequentially cancel the contribution of pivot rows of W to v
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for i ← 1 to N do
    if v = 0 then return 0
    if v_i ≠ 0 then
        if w_{i,i} ≠ 0 then v ← v − (v_i / w_{i,i}) w_i
                else return v
end for
```

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- Construct $W \oplus \mathrm{Span}(g)$, for $g \in \mathcal{G} \setminus W$:
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  - let i be the iteration at which the algorithm returns $\tilde{g}$ $(\neq 0)$
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  - then $w_i \leftarrow \tilde{g}$

- Improvement: since W is constructed incrementally
  - keep the vectors of $\mathcal{G}$ reduced by W at all times
  - when expanding W by $\mathrm{Span}(g)$, reduce $\mathcal{G}$ by g only

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Computing $\mathbf{W} \cap \mathcal{G}$

- Find all $g \in \mathcal{G}$ such that $g \in W$
- For each $g \in \mathcal{G}$, test if $\texttt{reduce}(g,\ W) = 0$?
  - slow (only one vector at a time, lots of conditional branches, etc.)
  - we don't need to reduce g by W

Computing $\mathbf{W} \cap \mathcal{G}$

- Find all $g \in \mathcal{G}$ such that $g \in W$
- For each $g \in \mathcal{G}$, test if $\texttt{reduce}(g,\ W) = 0$?
  - slow (only one vector at a time, lots of conditional branches, etc.)
  - we don't need to reduce g by W
- Idea (courtesy of E. Thomé): put matrix W in reduced row echelon form

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{j_1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{j_2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{j_r,j_r} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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  - all pivot columns j_i to 0, except for $w_{j_i,j_i} = 1$

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  - all pivot columns j_i to 0, except for $w_{j_i,j_i} = 1$
  - $g \in W$ if and only if $g^{\mathsf{T}} \cdot W' = g$

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  - $g \in W$ if and only if $g^\top \cdot W' = g$
  - batch 64, 128, or 256 products $g^\top \cdot (W' - \mathrm{Id})$